

APPROXIMATING THE STATE MANIFOLD GEOMETRY

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From Fisher metric to quantum state manifolds

The space of probability distributions $p(\Lambda)$ for $\Lambda \in \mathbb{R}^D$ can be equipped with a *Fisher information metric*. In quantum physics, we obtain probability distributions by taking $\langle A_i | \psi \rangle$ for a wave function $|\psi\rangle$ and eigenstate $|A_i\rangle$ of some observable \hat{A} . As a result, we get a **Projective Hilbert space** $\mathbb{C}P^n$ equipped with a *Fubini-study metric* [1]. Such a metric can be defined between two states $|\psi\rangle, |\phi\rangle \in \mathbb{C}P^n$ as $ds_{FS}^2 := \arccos |\langle \psi | \phi \rangle|$. Approximating for close states, we get a **Provost-Vallee (PV) metric** [2]

$$ds_{PV}^2 = 1 - |\langle \psi | \psi + \delta\psi \rangle|^2. \quad (1)$$

State manifolds

Consider a n -dimensional projective Hilbert space $\mathbb{C}P^n$ for $n \in \mathbb{N} \cup \{+\infty\}$, and Hamiltonian $\hat{H}(\Lambda)$ with *driving parameters*, denoted collectively as a vector $\Lambda \in \mathbb{R}^D$. The spectrum is assumed to be bound and non-degenerate, except for a finite number of points. The Schrödinger equation then reads $\hat{H}|\psi_n(\Lambda)\rangle = E_n|\psi_n(\Lambda)\rangle$.

The mapping $\Lambda \rightarrow |\psi_0(\Lambda)\rangle$ defines a **ground state manifold** (or any excited manifold analogically)

$$\mathcal{M}_0 := \cup_{\Lambda \in \mathbb{R}^D} |\psi_0(\Lambda)\rangle \quad (2)$$

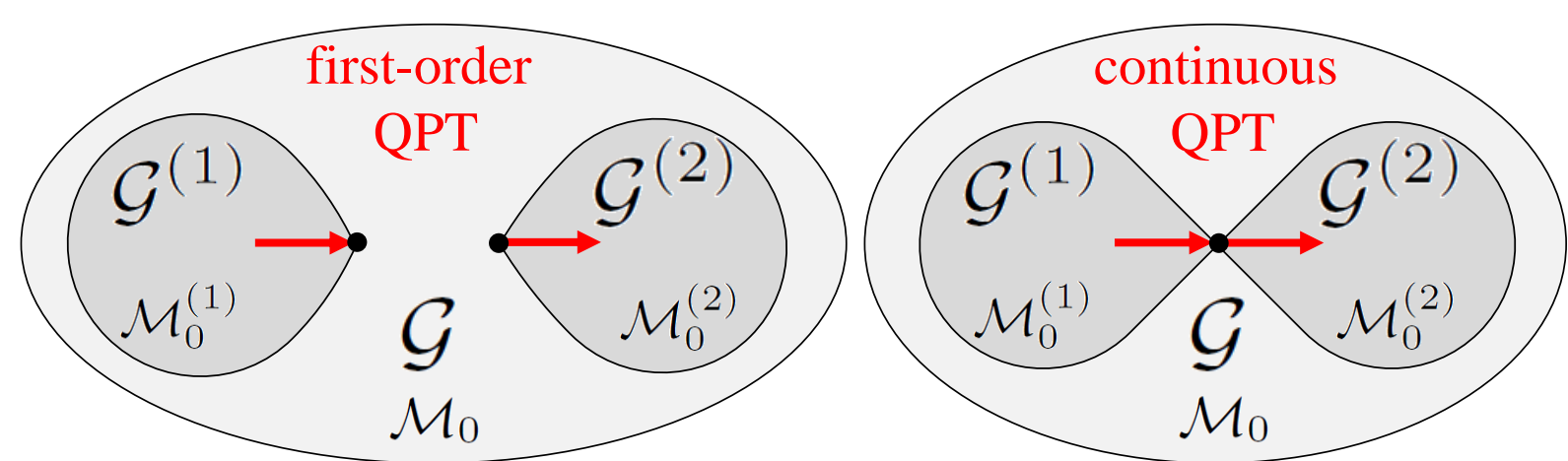
equipped with a **Geometric tensor**

$$Q_{\mu\nu} := \langle \psi_{0,\mu} | (\text{Id} - |\psi_0\rangle\langle\psi_0|) | \psi_{0,\nu} \rangle, \quad (3)$$

where comma denotes a partial derivative. A pullback of the **PV-metric** in Eq. 1 is then $g_{\mu\nu} = \text{Re}Q_{\mu\nu}$. In addition we get a symplectic form $\chi_{\mu\nu} = \text{Im}Q_{\mu\nu}$. The physical meaning of this geometry is still not fully understood. There are many special practical showcases, such as in *decohering driving* [3], or in quantum search algorithms [4]. Other implications for quantum state driving are still being discovered.

Quantum Phase Transitions

We distinguish between first-order QPT, where the wave function $|\psi_0\rangle$ is discontinuous and second-order, where some of its derivatives over Λ contain discontinuity.



Ground state manifold \mathcal{M}_0 consists of parts $\mathcal{M}_0^{(1)}$, $\mathcal{M}_0^{(2)}$ on both sides of a QPT. For the first-order QPT they may have incompatible invariant geometries $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$. G.s. manifolds for a continuous QPT are joined, so the system can be continuously driven over QPT.

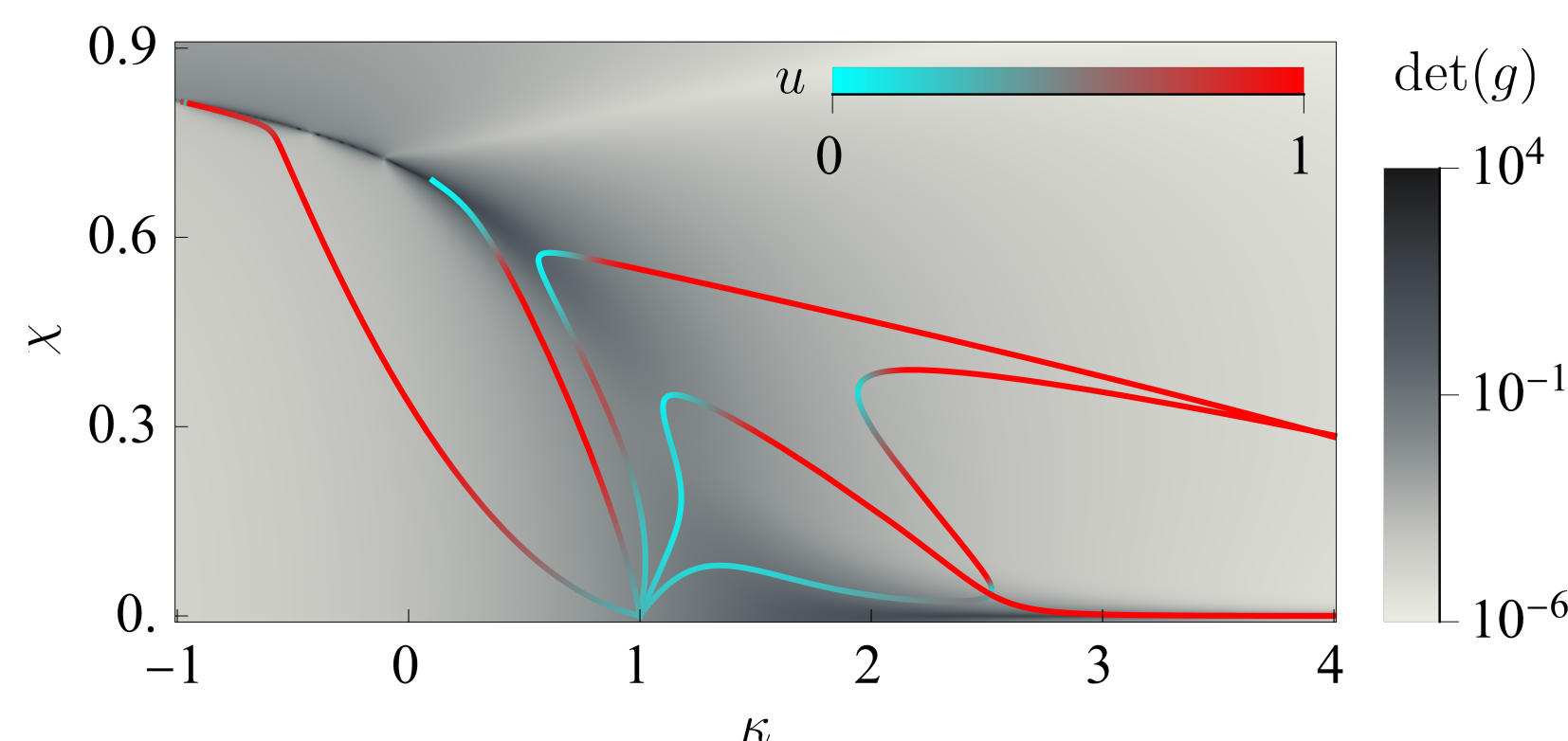
We demonstrate our findings on a fully-connected model consisting of two types of bosons s, t

$$\hat{H}^{(N)}(\kappa, \chi) = \frac{1}{2}(\hat{t}^\dagger \hat{t} - \hat{s}^\dagger \hat{s}) - \frac{1}{N} \left[\chi^2 \hat{t}^\dagger \hat{t} + \frac{\chi}{2} (\hat{t}^\dagger \hat{s} + \hat{s}^\dagger \hat{t}) \right] - \frac{1}{N} \left[\frac{\kappa}{4} (\hat{t}^\dagger \hat{t} \hat{s} \hat{s} + \hat{s}^\dagger \hat{s} \hat{t} \hat{t} + 2 \hat{t}^\dagger \hat{s} \hat{s} \hat{t}) + \chi (\hat{t}^\dagger \hat{t} \hat{s} \hat{t} + \hat{t}^\dagger \hat{s} \hat{t} \hat{t}) + \chi^2 \hat{t}^\dagger \hat{t} \hat{t} \right],$$

for energy scaling ϵ and $\kappa, \chi \in \mathbb{R}$.

Geometry of ground state manifold

The geometry of state manifolds is generally fairly complex and can be investigated using metric determinant, Ricci scalar or geodesics. Below is an example of the geodesic structure of our fully-connected model. The problem we aim to address is the breakdown of numerics in the vicinity of diabolic points (points Λ_i of spectrum degeneracy).



Geodesics on the background of the metric tensor determinant for the ground-state manifold of the N bosonic fully-connected model. Several general features are demonstrated: (i) The plain speed u in (κ, χ) (expressed in a relative scale of individual geodesics) is reduced with $\det(g)$. (ii) The geodesics have a tendency to be repelled or attracted from/to the diabolic points, see $(\kappa, \chi) = (-\frac{1}{9}, \sqrt{\frac{10}{19}})$.

Mean field approximation

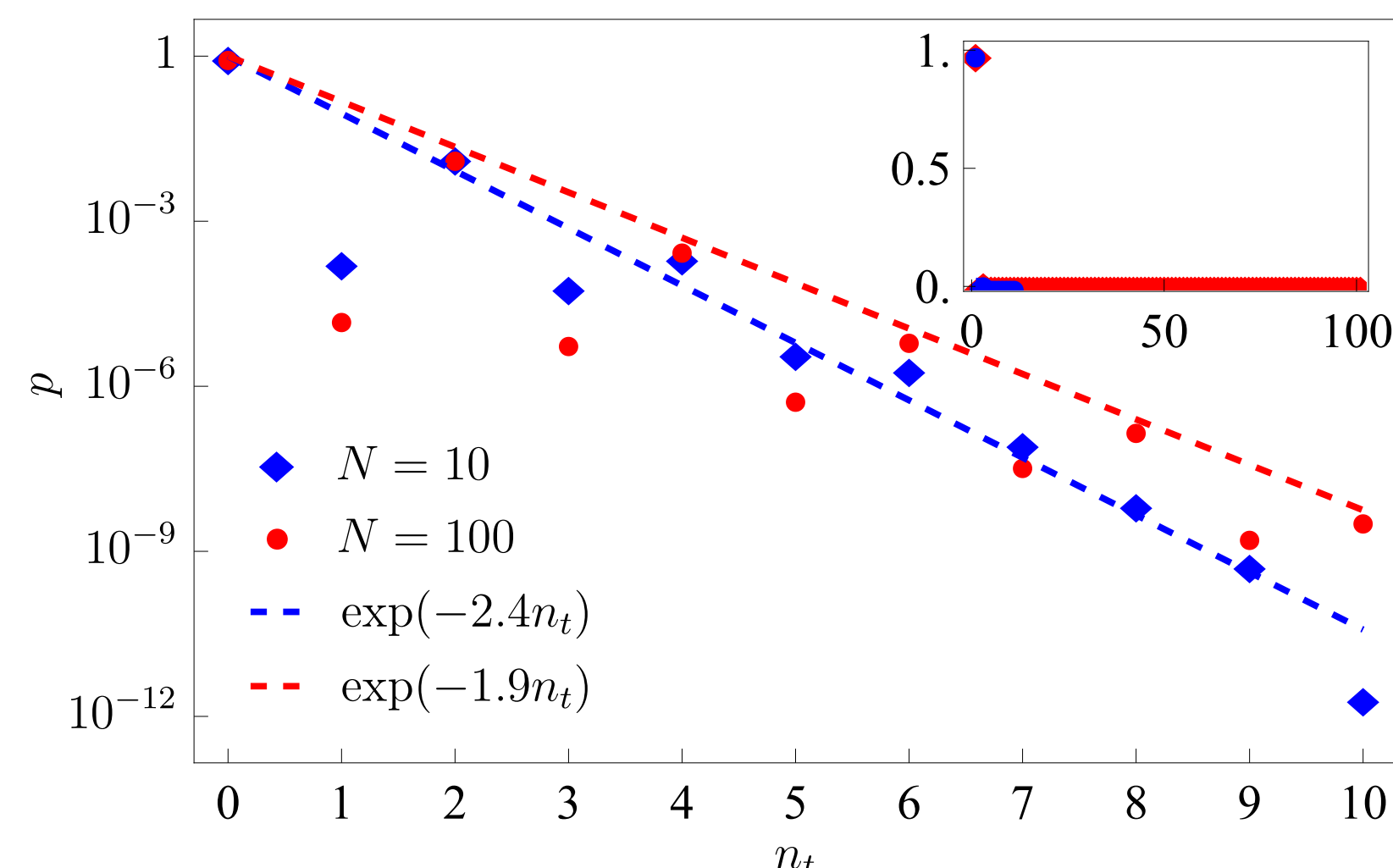
Can we use the mean-field to approximate the ground state manifold geometry? Not so much. In the mean-field approximation, we assume the ground state to be a condensate in which all bosons occupy the same single-particle state [5]. For systems composed of N bosons of K types, the condensate has the general form

$$|\psi_{MF}^{(N)}(\alpha)\rangle = \frac{1}{\sqrt{N!}} [\hat{B}(\alpha)^\dagger]^N |0\rangle, \quad (4)$$

$$\hat{B}(\alpha)^\dagger = \frac{1}{\sqrt{\mathcal{S}(\alpha)}} \left(\hat{b}_0^\dagger + \sum_{k=1}^{K-1} \rho_k e^{i\phi_k} \hat{b}_k^\dagger \right), \quad (5)$$

for the vacuum state $|0\rangle$, $\rho_k \in [0, \infty)$, $\phi_k \in [0, 2\pi)$, normalization factor \mathcal{S} and operator B^\dagger creating bosons to the condensate single-particle state determined by complex parameters $\alpha = (\alpha_0, \dots, \alpha_{K-1})$.

In our example, there exists a phase containing only s -bosons. As N increases, the ground state converges to this only approximately, but will always contain a finite amount of t -bosons.



Probability of measuring the \hat{H} ground state vector in a state with n_t t -bosons and $n_s = N - n_t$ s -bosons, i.e. $p = |\langle n_s, n_t | \psi_0^{(N)}(\Lambda) \rangle|^2$. Here $\Lambda = (0, 0.2)$ and $N \in \{10, 100\}$ are shown.

It might seem insignificant if $|\langle \psi_0 | \psi_{MF} \rangle|^2 \approx 10^{-6}$, but this leads to the *relative error in metric tensor elements* ≈ 0.2 and generally different invariant geometry.

Two level approximation

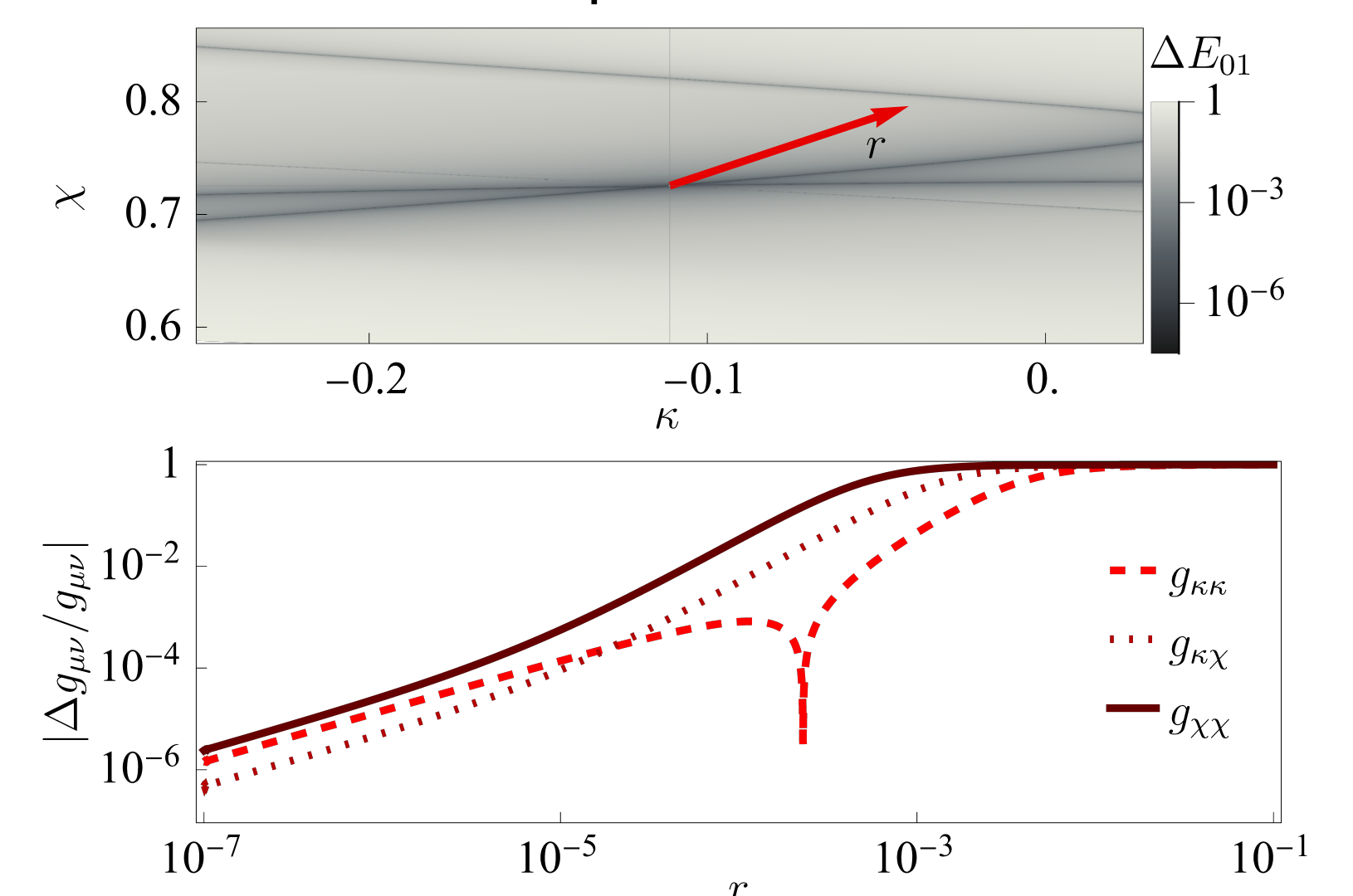
We can arrive to a better approximation by Taylor expansion of $\hat{H}(\Lambda)$ around diabolic point with coordinate Λ^D into 2. order (the order in Λ powers of the model)

$$\hat{H}_2(\Lambda) = \hat{H}(\Lambda^D) + \frac{\partial \hat{H}}{\partial \Lambda_\mu} \Big|_{\Lambda^D} (\Lambda_\mu - \Lambda_\mu^D) + \frac{1}{2} \frac{\partial^2 \hat{H}}{\partial \Lambda_\mu \partial \Lambda_\nu} \Big|_{\Lambda^D} (\Lambda_\mu - \Lambda_\mu^D) (\Lambda_\nu - \Lambda_\nu^D). \quad (6)$$

We can choose any basis $\{|0\rangle, |1\rangle\}$ of the degenerate subspace at DP and define

$$\hat{H}_{eff} \equiv \begin{pmatrix} \langle 0 | \hat{H}_2(\Lambda) | 0 \rangle & \langle 0 | \hat{H}_2(\Lambda) | 1 \rangle \\ \langle 1 | \hat{H}_2(\Lambda) | 0 \rangle & \langle 1 | \hat{H}_2(\Lambda) | 1 \rangle \end{pmatrix} \quad (7)$$

Such a Hamiltonian approximates the two lowest energy levels of the original \hat{H} . In addition, it provides a good approximation of the metric tensor elements in the vicinity of DP; see the relative error below. However, their derivatives along any direction in parametric space are not guaranteed to converge to the original model, leading to different invariant geometry. Any two-level system will have Ricci scalar $R = 8$ (sphere for $\Lambda \in \mathbb{R}^2$), but the original system has $R = R(\Lambda)$ and different limit as Λ approaches DP from both sides of QPT precursor.



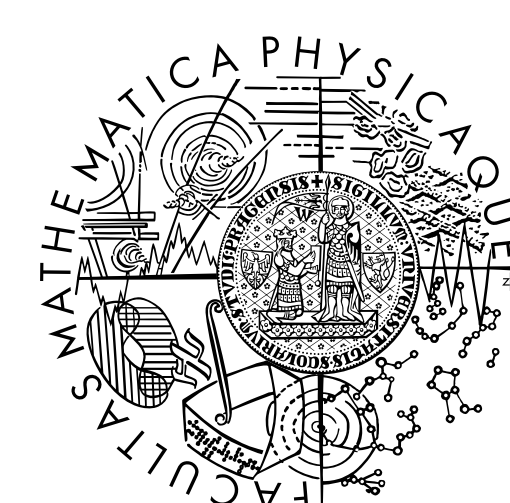
The approximation of energy gap $\Delta E_{01} = |E_{01} - E_{01}^{eff}|$ in the vicinity of DP for $N = 10$. The metric tensor elements are shown along the marked line from the DP at the angle $\pi/4$ in a radial direction $r \equiv |\Lambda - \Lambda_{DP}|$. The energy difference error as well as metric component relative error $|\Delta g_{\mu\nu} / g_{\mu\nu}|$ goes polynomially to zero as $r \rightarrow 0$.

Conclusions

We illustrate how first-order QPTs separate the ground state manifold into two non-connectable regions, while second-order QPTs allow for a smooth transition between geometric regimes. The approximations of the ground state manifold using mean-field fails due to a finite number of boson excitations, leading to significant errors in the geometric description. Further, we demonstrated how two-level systems geometry is fundamentally spherical and fails to reproduce the original manifold's invariant geometry. However, the two-level approximation can provide accurate metric tensor components locally near the degeneracy point. A faithful representation of the manifold geometry, thus requires going beyond these simplifying approximations. *The work is supported by Charles University GA UK 2023 No. 207723. and No. 215323.*

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